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Magnetohydrodynamic flows of non-equilibrium plasmas

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The present paper studies the interaction between non-equilibrium ionization phenomena in a plasma and the non-uniform flow of that plasma. The characteristics of non-equilibrium conditions are reviewed for the uniform flow situation in which electric and magnetic fields may exist. In order to permit study of the non-uniform flow situation a single-fluid theory is proposed in which the electrical conductivity is assumed to be a function of the absolute magnitude of the current density. The resulting highly simplified system of equations are then used to formulate the Hartmann-flow problem. For this case the formerly linear problem becomes non-linear. Despite this, solutions to within a quadrature are obtained when the Hall effect is neglected. The more complex problem including the Hall effect is solved numerically. Solutions showing velocity and current profiles are given. For certain values of the parameters governing the problem it is shown that no solutions exist.

1. Introduction

The principal objective of the present study is to examine the coupling between non-equilibrium ionization in a plasma and the non-uniform flow of that plasma. By non-equilibrium ionization we will mean a condition in a plasma wherein the electrons and heavy particles are at different average energies or temperatures, and the degree of ionization is greater than would exist at the gas temperature under thermal ionization. Since this non-equilibrium state depends on the local electric field (neglecting finite recombination and ionization rates), and this local field $\mathbf{E}_{\text{induced}}$ depends on the flow velocity through the term $\mathbf{v} \times \mathbf{B}$ in Ohm's law, it is apparent that some coupling will occur in a non-uniform flow.

As is well known, an electric field applied to a static plasma will raise the electron temperature and increase the degree of ionization. The energy lost in an elastic collision between an electron and heavy atom can be expressed as

$$\Delta\epsilon = 2(m_e/m_a)\,(\tfrac{3}{2}kT_e - \tfrac{3}{2}kT), \tag{1}$$

where T is the bulk gas temperature, T_e is the electron temperature, k is Boltzmann's constant, and m_e/m_a is the electron- to atom-mass ratio. For higher electron temperatures an appreciable fraction of collisions may be inelastic, so that photon emission due to de-excitation of the resulting excited states may constitute an additional energy loss if the radiation is not trapped. For the time being we shall assume that all such radiation is trapped so that (1) represents all of the energy lost. If we then multiply the above by the electron density n_e and the electron collision frequency ν_e, the resulting product represents the total energy loss per unit volume and time. Equating this to the energy transferred into the plasma by the electric field $\mathbf{j}.\mathbf{E}$ we have

$$\mathbf{j}.\mathbf{E} = n_e \nu_e \left(\frac{2m_e \delta}{m_a}\right)(\tfrac{3}{2}kT_e - \tfrac{3}{2}kT), \tag{2}$$

where δ is a constant equal to one for elastic collisions but greater than one for inelastic collisions with polyatomic molecules in which rotational and vibrational degrees of freedom can be excited, and $\mathbf{j}$ is the current density.†

From an expression of this sort, some estimate of the electron temperature may be obtained provided the state of the plasma is known. However, the state of the plasma is not known, since n_e depends on T_e as well as the gas conditions. The functional relationship between n_e and T_e is given by the Saha relation evaluated at the electron temperature rather than the gas temperature. Thus, we have

$$\frac{n_e^2}{n_0} = \frac{2g_A^+}{g_A}\left(\frac{2\pi m_e kT_e}{h^2}\right)^{\frac{3}{2}} \exp(-I/kT_e), \tag{3}$$

where n_0 is the neutral atom density, I the ionization potential, g the statistical weight, and h the Planck constant. The approximations inherent in using the Saha equation in this way, as well as the energy balance of (2), are discussed in Sherman (1964).

When a magnetic field exists as well as a flow, (2) still applies locally within the plasma if $\mathbf{E}$ is interpreted to be the electric field in the frame moving with the local fluid velocity $\mathbf{E}^* = \mathbf{E} + \mathbf{v} \times \mathbf{B}$.

Having established the physical nature of the non-equilibrium ionization phenomenon, and the method of describing it locally in a plasma, it is necessary to turn our attention to the equations describing the corresponding flows. From the Boltzmann equation the species mass, momentum, and energy conservation equations can be derived, and from these one can formulate a single-fluid theory which still permits inclusion of the Hall effect and non-equilibrium ionization (cf. Sherman 1964).

We first have the equations for mass and momentum conservation,

$$(\partial\rho/\partial t) + \nabla.(\rho\mathbf{v}) = 0, \tag{4}$$

and

$$\rho(D\mathbf{v}/Dt) + \nabla p = \mathbf{j} \times \mathbf{B} + \mu\nabla^2\mathbf{v}, \tag{5}$$

where ρ is the mass density, p the plasma pressure, and μ the plasma viscosity, and we are neglecting the electric body force as well as current flow due to the transport of excess charge.

It should be noted that, although the electron temperature has not entered the momentum equation explicitly, we have not yet related $\mathbf{j}$ to any of the other variables. When we do this we shall find an electrical conductivity entering the equations. It will be in the magnitude of this conductivity that the electron temperature will play its essential role and couple to (5).

† We must observe here that (2) is a simplified version of the complete partial differential equation governing the electron energy. In its present form (2) neglects electron energy convection, conduction, flux due to diffusion of electron enthalpy, and viscous dissipation.

When the species-momentum equations for electrons, ions and atoms are simplified by assuming the ions move at the atom velocity, and neglecting gradients in electron partial pressure, a 'generalized' Ohm's law can be derived (Cowling 1956):

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B} - (1/en_e)\,\mathbf{j} \times \mathbf{B}), \tag{6}$$

where $\sigma = e^2 n_e \tau / m_e$ is the plasma conductivity, τ being the collision time. Clearly, since n_e and τ depend on T_e, we find σ to be a function not only of the overall properties of the gas but of T_e as well.

To complete the formulation of the single-fluid equations, one needs a partial differential equation governing the energy of the plasma as a whole. In the present analysis, however, our principal interest will be in the coupling between the non-equilibrium ionization and the non-uniform flow. We shall, accordingly, omit the overall energy equation and assume the flow incompressible. In doing this we are, of course, neglecting the influence of overall plasma heat transfer on the magnetohydrodynamic flow. However, this assumption is quite necessary at this stage of our knowledge, as the complete problem would be much too complex. None the less, it must be remembered that we are not neglecting *all* effects of energy transfer since we are still allowing electrons to gain energy in the electromagnetic field and lose it in collisions with heavy particles.

Within the above assumptions, a *simplified* single-fluid theory can be obtained by assuming the electrical conductivity in (6) to be a function of the current density alone:

$$\sigma = \sigma(|\mathbf{j}|). \tag{7}$$

At the same time we assume the coefficient of the last term in (6), that is σ/en_e, to be constant, which is essentially the same as assuming a constant $\omega\tau = \sigma B/en_e$.

Making both of the above assumptions allows us to use a single-fluid theory to describe non-equilibrium ionization where the only new feature is to allow σ to be a function of $\mathbf{j}$.

The basis for assuming $\sigma = \sigma(|\mathbf{j}|)$ and $\omega\tau$ constant will be treated in the next section.

2. Conductivity approximation

If we neglect flow variations with position or time (2) gives us a relation between the electron temperature and the electric field. Thus

$$\mathbf{j}\,.\,\mathbf{E}^* = 3n_e \nu_e k(m_e/m_a)(T_e - T),$$

assuming $\delta = 1$. Using Ohm's law (6) to eliminate $\mathbf{E}^*$, we can then write

$$j^2/\sigma = 3n_e \nu_e k(m_e/m_a)(T_e - T). \tag{8}$$

However, from the simple kinetic theory we know

$$\sigma = e^2 n_e / m_e \nu_e, \tag{9}$$

so that combining (8) and (9) we have

$$j^2 = 3e^2 n_e^2 k(T_e - T)/m_a, \tag{10}$$

which is one relation between the electron temperature, the current density, and the electron density. The other relation between electron density and electron temperature is the Saha equation (3). By substituting n_e from (3) into (10) we get

$$j^2 = \frac{3e^2k}{m_a}(T_e - T)\frac{2g_A^+}{g_A}n_0\left(\frac{2\pi m_e kT_e}{h^2}\right)^{\frac{3}{2}}\exp(-I/kT_e). \tag{11}$$

In principle then we could solve (11) for $T_e = T_e(|\mathbf{j}|)$. Then using this relation in (8) we could obtain $n_e = n_e(|\mathbf{j}|)$. Finally, substituting $n_e = n_e(|\mathbf{j}|)$ into the conductivity expression (9), and since $\nu_e = \nu_e(T_e)$, substituting also from $T_e = T_e(|\mathbf{j}|)$ we can in principle obtain an expression for σ as a function of the absolute magnitude of the current density. Thus

$$\sigma = \frac{e^2 n_e(|\mathbf{j}|)}{m_e \nu_e(T_e)} = \frac{e^2 n_e(|\mathbf{j}|)}{m_e \nu_e(|\mathbf{j}|)} = \sigma(|\mathbf{j}|).$$

Also, for weakly ionized gases σ is essentially proportional to n_e, and we have for a uniform magnetic field

$$\omega\tau = \sigma B/en_e = \text{const.}$$

In practice $\omega\tau$ is not precisely a constant. However, it is a much less sensitive function of the non-equilibrium state than σ is, so that this should be a reasonable approximation.

Next, we consider the important question as to whether or not a *simple* functional dependence may be chosen for $\sigma(|\mathbf{j}|)$. Based on recent experimental measurements (Zukoski, Cool & Gibson 1964) shown in figure 1 it would seem reasonable for current densities greater than 1·0 amp/cm² to assume

$$\sigma = d|\mathbf{j}|^n \quad (0 < n < 1).$$

On the other hand, for very low current densities it would seem reasonable to assume (cf. Sherman 1964)

$$\sigma = \sigma_0 + c|\mathbf{j}|.$$

Thus, we are now in a position to formulate the channel-flow problem with a simple $\sigma(|\mathbf{j}|)$ relation, and look for solutions with a minimum of complexity.

3. Hartmann flow

Having evolved a sufficiently simple system of equations for the description of non-equilibrium phenomena in a magnetohydrodynamic flow, and having discussed the validity of the simplification, we next turn our attention to the flow problem using these equations.

Since the physical phenomena are of principal interest in the present study, it is appropriate to select the simplest flow problem which still retains the features to be studied. Toward this end we have chosen the problem of incompressible flow in a constant-area channel for further study. In the absence of magnetohydrodynamic effects this is usually referred to as Poiseuille flow. With a magnetic field applied but no unusual phenomena present, it is called the Hartmann flow. Then, if the Hall effect is included in the formulation, the problem can still be solved analytically (Sherman & Sutton 1961). That is, for all of the flows mentioned the equations, which include viscous effects, are linear so that simple solutions are possible.

The present work will treat the Hartmann flow when σ is a function of $|\mathbf{j}|$. The analysis will proceed in two steps. In the first, the Hall effect will be neglected since it is only in this case that simple solutions will be possible. The second will consider the full problem, with Hall effect, which must be solved numerically.

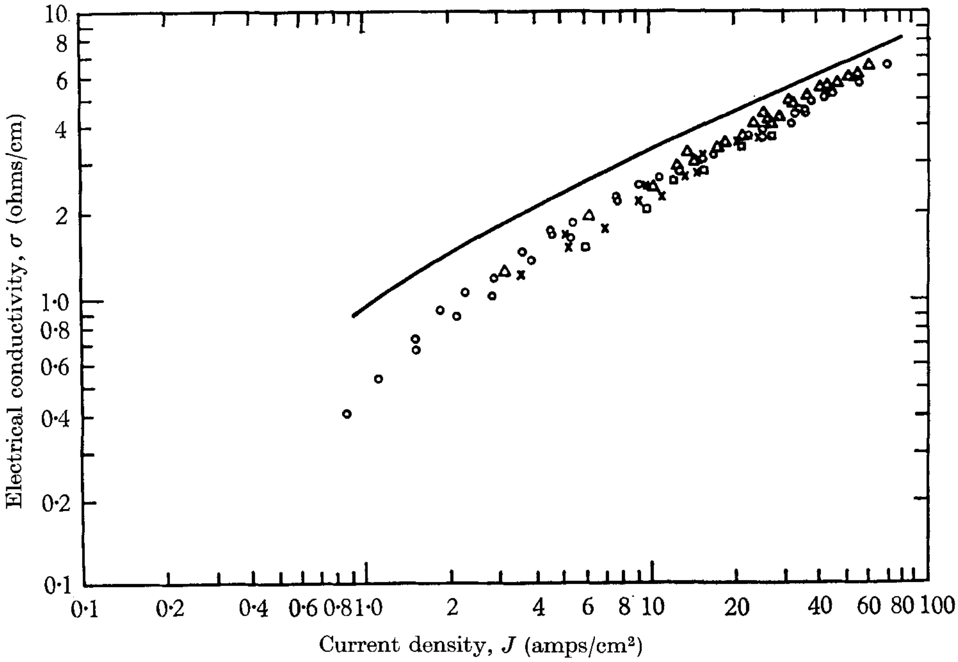

FIGURE 1. Electrical conductivity as a function of current density for a seeded argon plasma (after Zukoski *et al.* 1964), showing the theoretical curve for a ratio of potassium n_K to argon n_A of 0·0045 at an argon temperature of 2000 °K. Experimental points: ○, $n_K/n_A = 0{\cdot}0047$; △, $n_K/n_A = 0{\cdot}0041$; □, $n_K/n_A = 0{\cdot}0042$; ×, $n_K/n_A = 0{\cdot}0042$.

The primary advantage of solving the problem in the absence of Hall effect is that qualitative indications of the behaviour of non-equilibrium plasma flows can be obtained from analytical solutions to serve as a guide for the more difficult solutions.

In general, if $E = kuB$, then one can show from (2) that

$$T_e/T = 1 + \tfrac{1}{3}\gamma(1-k)^2(\omega\tau)^2 M^2,$$

where M is the Mach number. Now, from other work (Sherman & Sutton 1961) we know that the flow is relatively little affected for values of $\omega\tau$ of one or less. Similarly, compressible effects may be neglected when M is 0·3 or less, so that significant increases in T_e above T only occur when $k \gtrsim 3$. In a generator $0 < k < 1$ so that our first problem when applied to a generator will not be strictly realistic. It is interesting to observe, however, that for an accelerator $k > 1$, and the first problem may have a real physical counterpart if $k > 3$. None the less, this first problem is primarily useful for the guidance it lends to the solution of the more complex case with Hall effect.

The channel geometry for the problem to be considered is shown in figure 2. In order to proceed with the analysis for the case in which the Hall effect is neglected, it is assumed that all variables are functions of z alone except for p, which may be a function of x. In addition, the following assumptions regarding the variables are made:

$$\mathbf{v} = (u, 0, 0), \qquad \mathbf{j} = (0, j_y, 0),$$
$$\mathbf{B} = (b_x, 0, B_0), \quad \mathbf{E} = (0, E_y, 0).$$

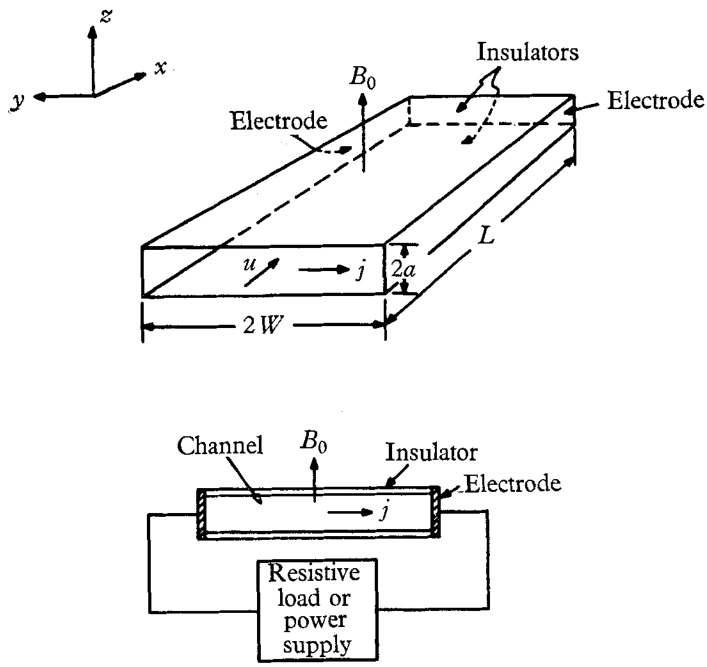

FIGURE 2. Channel geometry and co-ordinate system for Hartmann channel flow.

It should be noted that in the present problem it would be simple to calculate the induced magnetic field since it is not coupled to the flow problem. As our principal interest is in the latter, we shall not do so.

The basic equations were developed earlier and are repeated here for convenience.

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{12}$$

$$\nabla . (\rho \mathbf{v}) = 0 \tag{13}$$

$$\rho(D\mathbf{v}/Dt) + \nabla p = \mathbf{j} \times \mathbf{B} + \mu \nabla^2 \mathbf{v}. \tag{14}$$

With the assumed form of the variables, these reduce to

$$\frac{\partial p}{\partial x} = j_y B_0 + \mu \frac{d^2 u}{dz^2}, \tag{15}$$

$$j_y = \sigma(E_y - u B_0), \tag{16}$$

where $\partial p/\partial x$ is assumed constant. For convenience we shall define the following dimensionless variables

$$P_x = \frac{\partial p/\partial x}{\rho u_0^2/a}, \quad Z = z/a, \quad J_y = j_y/\sigma_0 u_0 B_0,$$
$$U = u/u_0, \quad k = E_y/u_0 B_0,$$

where u_0 is the flow velocity at the centre of the channel.

With these new variables (15) and (16) become

$$d^2U/dZ^2 = R_e P_x - H_a^2 J_y, \tag{17}$$

and

$$J_y = (\sigma/\sigma_0)(k-U), \tag{18}$$

where $R_e = \rho u_0 a/\mu$, $H_a^2 = \sigma_0 B_0^2 a^2/\mu$. The two alternative forms for the functional dependencies of $\sigma(|\mathbf{j}|)$ were noted earlier. They were

$$\sigma = d|j_y|^n, \quad \text{and} \quad \sigma = \sigma_0 + c|j_y|.$$

In dimensionless form they become

$$\sigma = d(\sigma_0 u_0 B_0)^n |J_y|^n = \sigma_R |J_y|^n, \tag{19}$$

and

$$\sigma/\sigma_0 = 1 + c'|J_y|, \tag{20}$$

where $c' = cu_0 B_0$. For the initial portion of this analysis we will restrict ourselves to the linear law (20).

Combining (20) with (18) we have

$$J_y = (1 + c'|J_y|)(k-U), \tag{21}$$

or, alternatively

$$J_y = (1 \pm c'J_y)(k-U), \tag{22}$$

where we choose the upper sign when $k > U$, the lower sign when $k < U$. Solving for J_y gives

$$J_y = \frac{k-U}{1 \mp c'(k-U)}. \tag{23}$$

It is interesting to observe that there is a limitation to the numerical value that c' may assume. If it is assumed that Ohm's law in the frame moving with the fluid must be obeyed, then the current density must have the same sign as the electric field $(k-U)$. However, in (23) we see that, whenever $c'|k-U| > 1$, we shall have J_y and $(k-U)$ of opposite signs. Since this is a contradiction of Ohm's law, we must restrict numerical values of c' to

$$c' < \frac{1}{|k-U|}.$$

On the other hand, when $c'|k-U| \sim 1$, the current density can be expected to approach infinity. The precise behaviour under these conditions will be clear from the solutions to be developed later. We can deduce, however, that attempts to apply or induce electric fields larger than $1/c'$ will cause an interaction with the flow which will destroy the steady laminar flow assumed here.

Returning to our equations we combine (17) and (23) to get

$$\frac{d^2U}{dZ^2} = R_e P_x - H_a^2 \left\{ \frac{k-U}{1 \mp c'(k-U)} \right\}, \tag{24}$$

where the boundary conditions are

$$U(\pm 1) = 0, \quad U(0) = 1, \quad U'(0) = 0.$$

The integration of this non-linear differential equation will depend, in part, on the range of U involved if $0 < k < 1$, since it changes its form when U is greater or less than k. Accordingly, the region of integration (treating U as the independent variable) must be split into $0 < U < k$ and $U > k$. We then have

$$\frac{d^2U}{dZ^2} = R_e P_x - H_a^2 \left\{ \frac{k-U}{1-c'(k-U)} \right\} \quad \text{for} \quad 0 < U < k, \tag{25a}$$

$$\frac{d^2U}{dZ^2} = R_e P_x - H_a^2 \left\{ \frac{k-U}{1+c'(k-U)} \right\} \quad \text{for} \quad k < U < 1. \tag{25b}$$

The integration of the first of these proceeds as follows:

$$\left(\frac{dU}{dZ}\right)^2 = 2R_e P_x U - 2H_a^2 \int \frac{(k-U)\,dU}{1-c'(k-U)}, \tag{26}$$

then

$$(dU/dZ)^2 = 2R_e P_x U + 2H_a^2 U/c' - 2(H_a^2/c'^2) \ln \{1-c'(k-U)\} + K_1. \tag{27}$$

The constant K_1 can only be evaluated when we know the value of dU/dZ for some value of U between 0 and k. The integration of the second equation proceeds in a similar manner. Thus

$$(dU/dZ)^2 = 2R_e P_x U - 2H_a^2 U/c' - 2(H_a^2/c'^2) \ln \{1+c'(k-U)\} + K_2. \tag{28}$$

For this case we can evaluate K_2 since we know that when $U = 1$ we have $dU/dZ = 0$. Thus,

$$K_2 = -2R_e P_x + 2(H_a^2/c')\,[1 + (1/c') \ln \{1 + c'(k-1)\}].$$

Now we are in a position to evaluate K_1 since we can calculate dU/dZ when $U = k$. Thus K_1 is found to be

$$K_1 = 2(R_e P_x - H_a^2/c')\,(k-1) - 2k(R_e P_x + H_a^2/c') + 2(H_a^2/c'^2) \ln \{1 + c'(k-1)\}.$$

With the constants K_1 and K_2 known, (27) and (28) become for $0 < U < k$

$$\frac{dU}{dZ} = \pm \left[2(R_e P_x - H_a^2/c')\,(k-1) + 2(U-k)\,(R_e P_x + H_a^2/c') - 2(H_a^2/c'^2) \ln \left\{ \frac{1-c'(k-U)}{1+c'(k-1)} \right\} \right]^{\frac{1}{2}}, \tag{29}$$

and for $k < U < 1$

$$\frac{dU}{dZ} = \pm \left[2(U-1)\,(R_e P_x - H_a^2/c') - 2(H_a^2/c'^2) \ln \left\{ \frac{1+c'(k-U)}{1+c'(k-1)} \right\} \right]^{\frac{1}{2}}. \tag{30}$$

The complete solution is then

$$Z+1 = \int_0^k \frac{dU}{\left[2\left(R_e P_y - \frac{H_a^2}{c'}\right)(k-1) + 2(U-k)\left(R_e P_x + \frac{H_a^2}{c'}\right) - \frac{2H_a^2}{c'^2} \ln \left\{ \frac{1-c'(k-U)}{1-c'(1-k)} \right\} \right]^{\frac{1}{2}}} + \int_k^U \frac{dU}{\left[2(U-1)\left(R_e P_x - \frac{H_a^2}{c'}\right) - \frac{2H_a^2}{c'^2} \ln \left\{ \frac{1-c'(U-k)}{1-c'(1-k)} \right\} \right]^{\frac{1}{2}}}. \tag{31}$$

As the integral now stands an integration in closed form has not been found. A numerical integration is not simple for two reasons. First, of the four parameters involved ($R_e P_x$, H_a^2, c', k) only three can be arbitrary since one is needed to satisfy the condition that $Z = 0$ when $U = 1$. When carrying out an analytical solution, this functional dependence can be determined once the dimensionless velocity profiles are calculated. For the present situation an iterative procedure must be followed. That is, with k, c' and H_a^2 fixed $R_e P_x$ must be varied until the integrals in (31) evaluated at $U = 1$ equal unity ($Z = 0$). Having carried out such an iterative procedure we have both the desired value of $R_e P_x$ for given k, c', and H_a^2 as well as the solution to the flow problem.

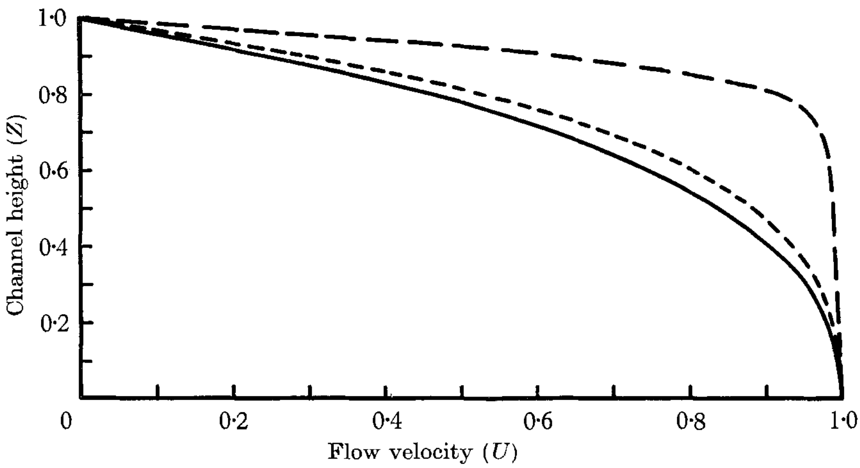

FIGURE 3. Axial velocity in the absence of the Hall effect for a short-circuited channel, $k = 0$, $H_a = 2$, as a function of the non-equilibrium parameter ($\sigma = \sigma_0 + c|\mathbf{j}|$ assumption). —, $c' = 0$, $R_e P_x = -5{\cdot}46$, and $c' = 0{\cdot}2$, $R_e P_x = -6{\cdot}43$; ----, $c' = 0{\cdot}5$, $R_e P_x = -8{\cdot}92$; — —, $c' = 0{\cdot}9$, $R_e P_x = -40{\cdot}0$.

The second difficulty is the singular nature of the second integral in (31) when $U \to 1$. The first question to be raised is, does this integral converge at all? Secondly, if it does converge, is it possible in general to find a value of $R_e P_x$ for which $Z = 0$ when $U = 1$? The first question can be answered in the affirmative by the use of the comparison test. The second question is more subtle. It has only been studied in detail for the $k = 0$ case. Considering (31) we recognize that solutions cannot exist for selected values of c' and H_a if $R_e P_x$ is such as to cause the argument of the square root to be negative for any value of U between zero and one. However, even if $R_e P_x$ is such as to cause the argument to be positive for all values of U, it is still not certain that Z will approach zero as $U \to 1$. Now, when $Z \to 0$ and $U \to 1$, we see from (31) that the integral from zero to one must be unity for a solution to exist. If the critical value of $R_e P_x$ (a negative number) required to keep the integrand just real also makes the integral from zero to one *less* than unity, then there can be no solution. This occurs, since making $R_e P_x$ more positive makes the integrand imaginary, while making it more negative *decreases* the value of the integral. However, a solution does exist if the critical value of $R_e P_x$ yields an integral from zero to one *greater* than or equal to unity.

By use of the comparison test one can verify that for some cases solutions do not exist. These are in fact those cases for which no calculations have been reported.

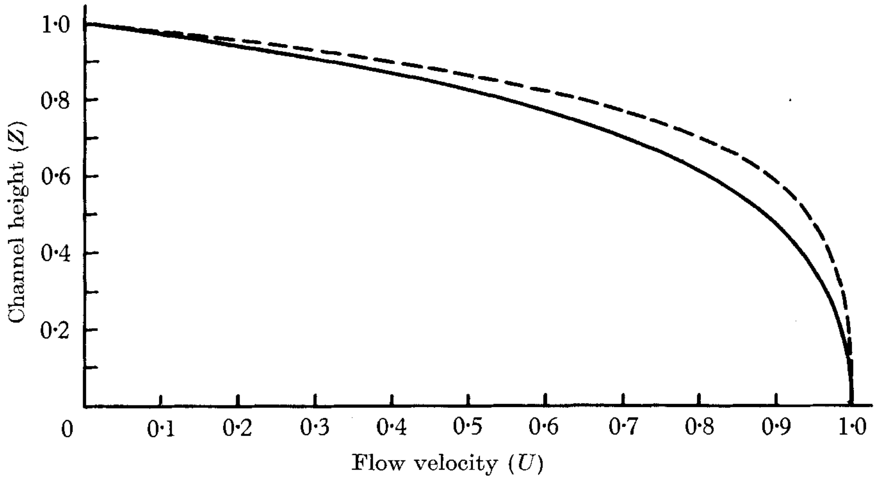

FIGURE 4. Axial velocity in the absence of the Hall effect for a short-circuited channel, $k = 0$, $H_a = 3$, as a function of the non-equilibrium parameter ($\sigma = \sigma_0 + c|\mathbf{j}|$ assumption). ——, $c' = 0$, $R_e P_x = -10{\cdot}0$, and $c' = 0{\cdot}2$, $R_e P_x = -12{\cdot}2$; – – – –, $c' = 0{\cdot}5$, $R_e P_x = -18{\cdot}4$.

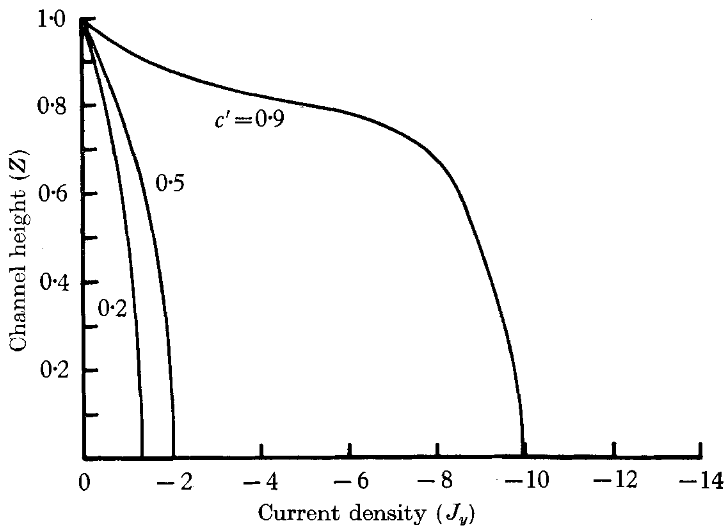

FIGURE 5. Transverse current in the absence of the Hall effect for a short-circuited channel with $k = 0$ and $H_a = 2$ as a function of the non-equilibrium parameter ($\sigma = \sigma_0 + c|\mathbf{j}|$ assumption).

Having verified the convergence of our integral we have carried out numerical integrations for several cases of interest. The velocity profiles for these cases are shown in figures 3 and 4, and the current density profile for the $H_a = 2$ case is shown in figure 5. The curves for $c' = 0$ were very close to the curves for $c' = 0{\cdot}2$, and no solution was possible for $H_a = 3$, $c' = 0{\cdot}9$.

In general, we find that, if $c' \sim 1$, the velocity profile becomes very full. Also, in all cases, increasing c' from zero tended to fill out the velocity profile.

From the current distributions we recognize the nature of the current density behaviour when $c' \to 1/|k-U|$. That is, as c' approaches this limit the velocity profile tends to become square and the current density tends to become infinite all across the channel.

The axial pressure gradients are a last point of interest. We observe that they become quite large as c' approaches its limiting value.

Turning our attention next to the case where

$$\sigma = \sigma_R |J_y|^n,$$

we can substitute this in (18). Then we get

$$J_y = (\sigma_R/\sigma_0)\,|J_y|^n\,(k-U). \tag{32}$$

To solve for J_y, we have to restrict U to less than or greater than k. Thus

$$J_y = (\sigma_R/\sigma_0)^{1/(1-n)}\,(k-U)^{1/(1-n)} \quad \text{for} \quad U < k, \tag{33}$$

$$J_y = -(\sigma_R/\sigma_0)^{1/(1-n)}\,(U-k)^{1/(1-n)} \quad \text{for} \quad U > k. \tag{34}$$

From this new Ohm's law we can again draw some interesting conclusions even before solving any flow problem. We note that now we have the two parameters σ_R and n specifying the non-equilibrium ionization phenomena rather than the one c' as before. We might anticipate that n rather than σ_R plays the major role. That it does is verified by our equations (33), (34) which show that, when $n = 1$, we find J_y either infinite or indeterminate so that this is our limiting value now. On the other hand when $n = 0$ we have $\sigma = \text{const.}$ and obtain a Hartmann flow so that n must lie between 0 and 1. If $\sigma_R = 0$ though, we have $\sigma \equiv 0$, the Lorentz force is zero, and we find a Poiseuille flow.

Proceeding to the solution we combine (33), (34) with (17) and integrate once to get

$$\left(\frac{dU}{dZ}\right)^2 = 2R_e P_x U - 2\hat{H}_a^2\left(\frac{1-n}{2-n}\right)(k-U)^{(2-n)/(1-n)} + K_1 \quad \text{for} \quad U < k, \tag{35}$$

$$\left(\frac{dU}{dZ}\right)^2 = 2R_e P_x U + 2\hat{H}_a^2\left(\frac{1-n}{2-n}\right)(U-k)^{(2-n)/(1-n)} + K_2 \quad \text{for} \quad U > k, \tag{36}$$

where
$$\hat{H}_a^2 = (\sigma_R/\sigma_0)^{1/(1-n)} H_a^2.$$

Clearly, when $n = \frac{1}{2}$ or $\frac{2}{3}$, we are led to elliptic integrals when we complete the solution. For other values of n we must complete the solution by numerical quadratures as before. As an illustration let us consider $n = \frac{1}{2}$. Using the same method as the previous case to determine K_1 and K_2 we find for $n = \frac{1}{2}$

$$K_1 = \tfrac{2}{3}\hat{H}_a^2(k-1)^3 - 2R_e P_x = K_2. \tag{37}$$

Integrating with respect to U we then obtain from (35) and (36)

$$dZ = \frac{dU}{[2R_e P_x(U-1) - \frac{2}{3}\hat{H}_a^2\{(k-U)^3 - (k-1)^3\}]^{\frac{1}{2}}} \quad \text{for} \quad U < k, \tag{38}$$

$$dZ = \frac{dU}{[2R_e P_x(U-1) - \frac{2}{3}\hat{H}_a^2\{(k-U)^3 - (k-1)^3\}]^{\frac{1}{2}}} \quad \text{for} \quad U > k. \tag{39}$$

Since (38) and (39) are identical, it is not necessary to restrict U so that the complete solution is

$$Z+1 = \int_0^U \frac{dU}{[2R_e P_x(U-1) - \frac{2}{3}\hat{H}_a^2\{(k-U)^3-(k-1)^3\}]^{\frac{1}{2}}}. \tag{40}$$

The integral can clearly be expressed as elliptic integrals and the solution is completed. Of course, iteration must be used to determine $R_e P_x$ as before.

In order to illustrate the procedure, we shall carry out calculations for the $k = 0$ case. In this instance (40) can be rewritten in a standard form for elliptic integrals. Thus

$$Z+1 = \frac{\sqrt{(\frac{3}{2})}}{\hat{H}_a}\int_0^U \frac{dU}{[(1-U)\{-(3R_e P_x/\hat{H}_a^2)-0{\cdot}75-(U+0{\cdot}5)^2\}]^{\frac{1}{2}}}.$$

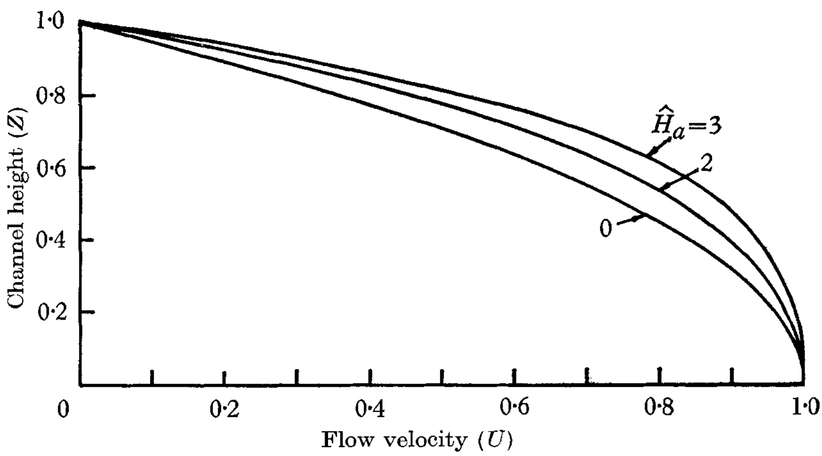

FIGURE 6. Axial velocity in the absence of the Hall effect for a short-circuited channel, $k = 0$, as a function of the non-equilibrium modified Hartmann number ($\sigma = \sigma_R|\mathbf{j}|^{\frac{1}{2}}$ assumption). For $\hat{H}_a = 0$, $R_e P_x = -2{\cdot}00$; $\hat{H}_a = 2$, $R_e P_x = -5{\cdot}20$; $\hat{H}_a = 3$, $R_e P_x = -9{\cdot}67$.

The solution is then

$$Z+1 = \frac{2\{F(\alpha,\phi[U=0])-F(\alpha,\phi)\}}{[\{-(3R_e P_x/\hat{H}_a^2)-0{\cdot}75\}^{\frac{1}{2}}+1{\cdot}5]^{\frac{1}{2}}},$$

where
$$F(\alpha,\phi) = \int_0^\phi \frac{d\phi}{\sqrt{(1-k^2\sin^2\phi)}}, \quad \sin\alpha = k,$$

and
$$\sin\alpha = \frac{2\sqrt{\{-(3R_e P_x/\hat{H}_a^2)-0{\cdot}75\}}}{\sqrt{\{-(3R_e P_x/\hat{H}_a^2)-0{\cdot}75\}}+1{\cdot}5},$$

$$\cos 2\phi = -\frac{U+0{\cdot}5}{\sqrt{\{-(3R_e P_x/\hat{H}_a^2)-0{\cdot}75\}}}.$$

Of course, $F(\alpha,\phi)$ is the elliptic function of the first kind and is a tabulated function.

Carrying out the mentioned iteration we find solutions for $H_a = 2, 3$. Curves of the resulting velocity and current density profiles are shown in figures 6 and 7. Now since $\hat{H}_a = (\sigma_R/\sigma_0)H_a$, where $\sigma_R = d\sqrt{(\sigma_0 u_0 B)}$ we can interpret an increasing value of $\hat{H}_a$ as an increase in the constant d with H_a constant. As one might have anticipated, increasing $\hat{H}_a$ makes the velocity profiles fuller. This could be interpreted either as due to an increase in the normal Hartmann parameter H_a or as due to a larger non-equilibrium effect, d. As before, when the parameter $\hat{H}_a$ increases and the velocity profiles become fuller, the pressure gradient becomes a larger negative number.

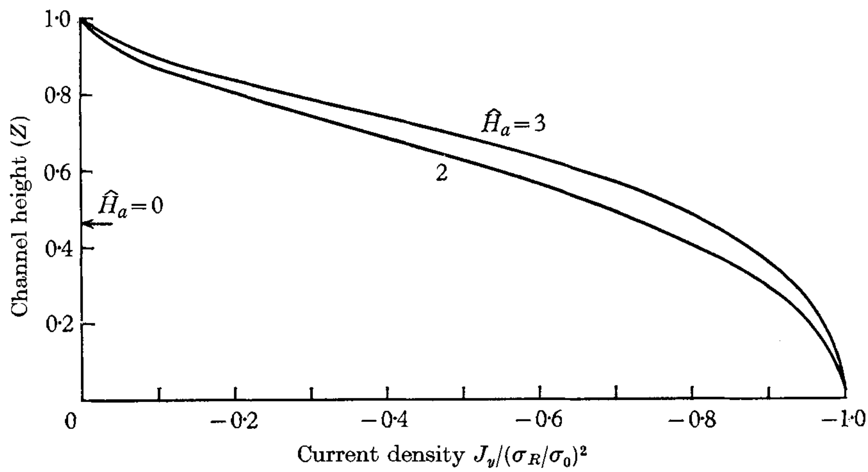

FIGURE 7. Transverse current in the absence of the Hall effect for a short-circuited channel, $k = 0$, as a function of the non-equilibrium modified Hartmann number ($\sigma = \sigma_R|\mathbf{j}|^{\frac{1}{2}}$ assumption).

4. Hartmann flow with Hall effect

Up to this point the Hall effect has been left out of the problem for simplicity. However, for most applications non-equilibrium ionization occurs most readily at lower pressures, and this is precisely the condition under which the Hall effect must be accounted for. The inclusion of both phenomena will now be considered.

First, it must be recognized that a cross-flow will exist (Sherman & Sutton 1961) so that $\mathbf{v} = (u, v, 0)$ and $\mathbf{j} = (j_x, j_y, 0)$. In this case the two components of the momentum equation become

$$\frac{\partial p}{\partial x} = j_y B_0 + \mu \frac{d^2 u}{dz^2}, \tag{41}$$

$$\frac{\partial p}{\partial x} = -j_x B_0 + \mu \frac{d^2 v}{dz^2}. \tag{42}$$

The principal difficulty will be in expressing j_x and j_y in terms of u and v as well as the other parameters of the problem. Now from our 'generalized' Ohm's law

$$\mathbf{j} = \sigma\{\mathbf{E} + \mathbf{v} \times \mathbf{B} - (\omega\tau/\sigma B_0)\,\mathbf{j} \times \mathbf{B}\}, \tag{43}$$

we can show readily

$$j_x = \frac{\sigma}{1+(\omega\tau)^2}\{E_x + vB_0 - \omega\tau(E_y - uB_0)\}, \tag{44}$$

$$j_y = \frac{\sigma}{1+(\omega\tau)^2}\{E_y - uB_0 + \omega\tau(E_x + vB_0)\}. \tag{45}$$

Assuming the linear law for $\sigma(|\mathbf{j}|)$, the generalization for the present problem is

$$\sigma = \sigma_0 + c\sqrt{(j_x^2 + j_y^2)}. \tag{46}$$

Substituting this into (44) and (45) and solving for j_x and j_y we get

$$j_x = \frac{\sigma_0\{(E_x + vB_0) - \omega\tau(E_y - uB_0)\}/\{1 + (\omega\tau)^2\}}{1 - c\{(E_x + vB_0)^2 + (E_y - uB_0)^2\}^{\frac{1}{2}}/\{1 + (\omega\tau)^2\}^{\frac{1}{2}}}, \tag{47}$$

$$j_y = \frac{\sigma_0\{(E_y - uB_0) + \omega\tau(E_x + vB_0)\}/\{1 + (\omega\tau)^2\}}{1 - c\{(E_x + vB_0)^2 + (E_y - uB_0)^2\}^{\frac{1}{2}}/\{1 + (\omega\tau)^2\}^{\frac{1}{2}}}. \tag{48}$$

Substitution of these expressions in (41) and (42) then gives us the desired two equations for u and v. If we introduce dimensionless variables as before, we get

$$\frac{d^2U}{dZ^2} = R_e P_x - \frac{H_a^2\{(k - U) + \omega\tau(\mathscr{E}_x + V)\}/\{1 + (\omega\tau)^2\}}{1 - c'\{(\mathscr{E}_x + V)^2 + (k - U)^2\}^{\frac{1}{2}}/\{1 + (\omega\tau)^2\}^{\frac{1}{2}}}, \tag{49}$$

$$\frac{d^2V}{dZ^2} = R_e P_y + \frac{H_a^2\{(\mathscr{E}_x + V) - \omega\tau(k - U)\}/\{1 + (\omega\tau)^2\}}{1 - c'\{(\mathscr{E}_x + V)^2 + (k - U)^2\}^{\frac{1}{2}}/\{1 + (\omega\tau)^2\}^{\frac{1}{2}}}, \tag{50}$$

and the boundary conditions are

$$U(\pm 1) = 0, \quad V(\pm 1) = 0,$$
$$U(0) = 1, \quad V'(0) = 0,$$
$$U'(0) = 0,$$

In order to complete the formulation of this problem, we note that we must require that

$$\int_0^1 J_x dZ = 0, \quad \int_0^1 V\,dZ = 0.$$

That is, there is neither any net current flow downstream nor any net mass flow cross-wise. As before, we shall select the parameters k, c', $\omega\tau$, H_a. The adjustable parameters are now $R_e P_x$, $R_e P_y$, $\mathscr{E}_x$. They are determined by the two integral conditions above, and the dV/dZ $(Z = 0)$ condition. Before discussing the solution of (49) and (50), we should comment on their form. First, when $c' \to 0$ the well-known Hartmann-flow problem is recovered. Also, as before, for some choice of c' the current density tends toward ∞, although now the permissible value is larger for larger $\omega\tau$ values. Finally, when $\omega\tau = \infty$, the non-equilibrium phenomena disappear, and we obtain the result that $j_x = 0$ and $j_y = \text{const.}$ (Sherman & Sutton 1961).

The solution of the coupled non-linear set of equations for U and V was carried out on an analogue computer, since we were faced with a two-point boundary-value problem with a threefold iteration. The read-out from the computer was placed on an oscilloscope, and the three arbitrary parameters were adjusted manually until the desired conditions were met. Using this technique, the cases given in table 1 were calculated for $H_a = 3$. Curves showing the velocity profiles for some of the cases are presented in figures 8 and 9. Based on these results a number of interesting observations can be made. First, as before, increasing c' increases the axial pressure gradient when $k = 0$, $0{\cdot}5$, and in some cases leads to no solutions being possible. Also, a transverse pressure gradient now exists to keep the net cross-wise flow zero. It is largest when the cross-flow velocity is greatest. As expected, an axial electric field exists to keep the net axial current flow zero. It is largest when $\omega\tau = 10$.

k	$\omega\tau$	c'	$R_e P_x$	$R_e P_y$	$\mathscr{E}_x$
0	1	0·5	−16·10	−0·704	−0·802
0	10	0·5	−11·30	−0·190	−6·72
0	10	0·9	−18·62	−0·314	−6·86
0·5	1	1	−7·17	−0·712	−0·277
0·5	10	1	−3·93	−0·147	−1·71
0·5	10	1·9	−4·44	−0·183	−1·74

TABLE 1

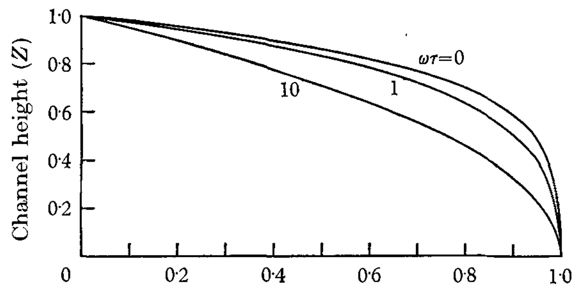

FIGURE 8

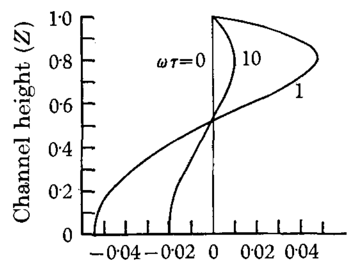

FIGURE 9

FIGURE 8. Axial velocity with the Hall effect for a short-circuited channel, $k = 0, H_a = 3$, $c' = 0{\cdot}5$, as a function of $\omega\tau$ ($\sigma = \sigma_0 + c|\mathbf{j}|$ assumption).

FIGURE 9. Transverse velocity with the Hall effect for a short-circuited channel, $k = 0$, $H_a = 3$, $c' = 0{\cdot}5$, as a function of $\omega\tau$ ($\sigma = \sigma_0 + c|\mathbf{j}|$ assumption).

From the flow profiles we note that the axial flow becomes more parabolic as $\omega\tau$ increases, which confirms our earlier results for the $c' = 0$ case (Sherman & Sutton 1961). Also, the transverse flow is minimized when $\omega\tau$ is larger. The differences between the $\omega\tau = 1$ and 10 cases have, however, been accentuated compared with when $c' = 0$. Accordingly, when c' is close to its limiting value, increasing $\omega\tau$ is more effective in limiting the influence of c' than it was before.

Finally, we consider the case where

$$\sigma = d\sqrt{|\mathbf{j}|} = d(j_x^2 + j_y^2)^{\frac{1}{4}}. \tag{51}$$

If we substitute this into (44) and (45) and solve for j_x and j_y, we find

$$j_x = \frac{d^2}{\{1+(\omega\tau)^2\}^{\frac{3}{2}}}\{(E_x + vB_0) - \omega\tau(E_y - uB_0)\}\{(E_x + vB_0)^2 + (E_y - uB_0)^2\}^{\frac{1}{2}}, \tag{52}$$

$$j_y = \frac{d^2}{\{1+(\omega\tau)^2\}^{\frac{3}{2}}}\{(E_y - uB_0) + \omega\tau(E_x + vB_0)\}\{(E_x + vB_0)^2 + (E_y - uB_0)^2\}^{\frac{1}{2}}. \tag{53}$$

Finally, if we substitute these in (41) and (42), we obtain two equations for U and V. They are

$$\frac{d^2U}{dZ^2} = R_e P_x - \frac{\hat{H}_a^2}{\{1+(\omega\tau)^2\}^{\frac{3}{4}}}\{(k-U)+\omega\tau(\mathscr{E}_x+V)\}\{(\mathscr{E}_x+V)^2+(k-U)^2\}^{\frac{1}{4}}, \quad (54)$$

$$\frac{d^2V}{dZ^2} = R_e P_y + \frac{\hat{H}_a^2}{\{1+(\omega\tau)^2\}^{\frac{3}{4}}}\{(\mathscr{E}_x+V)-\omega\tau(k-U)\}\{(\mathscr{E}_x+V)^2+(k-U)^2\}^{\frac{1}{4}}. \quad (55)$$

The boundary conditions are the same for this problem and the solution was carried out in the same way. The cases considered are given in table 2, and

k	$\omega\tau$	$\hat{H}_a$	R_eP_x	R_eP_y	$\mathscr{E}_x$
0	1	3	−8·43	−0·360	−0·778
0	10	3	−6·22	−0·085	−6·76
0·5	1	3	−2·94	−0·165	−0·228
0·5	10	3	−2·30	−0·020	−1·70

TABLE 2

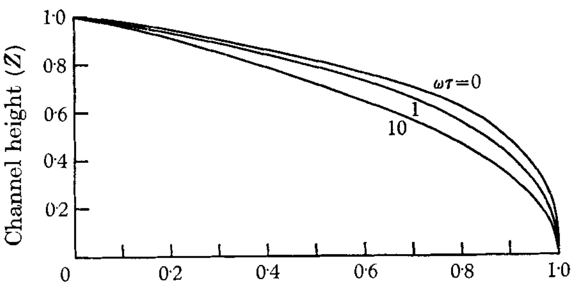

FIGURE 10

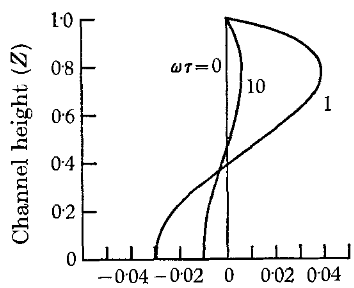

FIGURE 11

FIGURE 10. Axial velocity with the Hall effect for a short-circuited channel, $k = 0$, $\hat{H}_a = 3$, as a function of $\omega\tau$ ($\sigma = \sigma_R|\mathbf{j}|^{\frac{1}{2}}$ assumption).

FIGURE 11. Transverse velocity with the Hall effect for a short-circuited channel, $k = 0$, $\hat{H}_a = 3$, as a function of $\omega\tau$ ($\sigma = \sigma_R|\mathbf{j}|^{\frac{1}{2}}$ assumption).

velocity profiles for some are shown in figures 10 and 11. As before in the linear case increasing $\omega\tau$ from 1 to 10 tends to make the velocity profile more parabolic. The cross-flow velocity profiles are similar to those obtained earlier although smaller in magnitude. Also, similar transverse pressure gradients and axial electric fields exist as before.

5. Concluding remarks

In the present paper the problem of evaluating the coupling between the non-equilibrium ionization effect and a non-uniform flow has been considered. After a review of the basic phenomena and the equations describing them it was concluded that, if the elevation of the electron temperature can be directly related to the current density in a first approximation, then a particularly simple set of equations may be used. In fact, the familiar magnetohydrodynamic equations may be used intact if we only allow σ to be a function $|\mathbf{j}|$.

The essential feature discovered by taking σ to be either linear in $|\mathbf{j}|$ or varying as a power of $|\mathbf{j}|$ is that the degree of non-equilibrium assumed is not arbitrary, but is in fact limited. Also, we find that the new Ohm's law derived by letting $\sigma = \sigma(|\mathbf{j}|)$ reduces the formerly linear channel-flow problem to a non-linear one.

Considering the two simplified assumptions for the $\sigma(|\mathbf{j}|)$ functional dependence, the Hartmann flow was first formulated without consideration of the Hall effect. Despite the non-linearity, we showed that this problem can be solved within a quadrature. The velocity profiles become fuller the greater the non-equilibrium effect and the axial pressure gradients increase.

Including the Hall effect yields cross-flows and a cross-wise pressure gradient. Just as when the non-equilibrium effect is omitted, higher Hall parameters lead to more parabolic axial velocity profiles and less cross-flows. Another interesting facet arises when the axial Hall voltage is considered. In the absence of the non-equilibrium effect the axial Hall potential should be $\omega\tau\langle u\rangle B_z$ (cf. Sherman & Sutton 1961). When the non-equilibrium effect is included, we then find the Hall potential *increased* by a few per cent (Sherman 1965).

Most important, however, we have been able to show that, with the single-fluid formulation, the coupling between the non-equilibrium effect and the flow can be illuminated by the solution of relatively simple problems.

One must also remember, however, that such simplification has been bought at the expense of numerous assumptions. The two most essential are the neglect of energy-transfer effects, and the omission of finite ionization and recombination rates.

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